

# Fundamental laws revisited: the $\cos^4$ law of radiometry

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The cosine-fourth-power law describes the illuminance distribution of an optical image and its characteristic darker image edges. Besides vignetting (if present), the main effect is due to the chief ray angle, but there are also contributions by distortion and pupil aberrations, and by the intensity distribution if the object.

We notice that the generic textbook derivation is insufficient for practical optical design work. We analyze the involved approximations, review the literature on the topic, and present various apparently similar optical designs with different image illuminance distributions.

## 1 Introduction

In the absence of vignetting, distortion, and pupil aberrations, we can calculate the illuminance distribution in the following way. An optical system is imaging a Lambertian object of field angle  $\theta$  (Fig. 1).

A cosine factor comes into play four times :

- tilt of an area element of the object seen from the entrance pupil
- tilt of the pupil as seen from a field point
- the pupil distance gets longer with respect to the axial value by  $\cos^2 \theta$ .

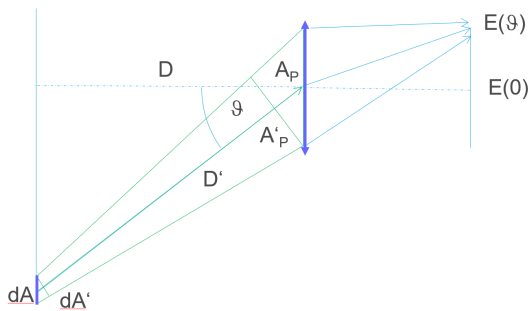


Fig. 1  $\cos^4$  law derivation

Hence the etendue of the light tube formed by an area element  $dA$  on the object and the pupil  $A_P$  is [1]

$$U(\theta) = \frac{dA \cdot A_P}{D^2} sr \cos^4 \theta \quad (1)$$

and the flux through the tube and subsequently to a  $dA'$  is with the object luminance  $L_O$

$$\Phi(\theta) = L_O \cdot U(\theta) \quad (2)$$

finally giving an image illuminance of

$$E'(\theta) = \frac{\Phi(\theta)}{dA'} = \frac{L_O A_P}{M^2 D^2} sr \cos^4 \theta \quad (3)$$

with the magnification  $M^2 = dA'/dA$ .

## 2 Illumination

According to the derivation, the  $\cos^4$  law is only valid for a Lambertian object as we make use of the constant luminance. It is not valid for e.g. Köhler illumination. For Lambertian objects, the image illuminance distribution (shown in a non-sequential run) corresponds well to the *relative illumination* of the optical design software.

## 3 Optical design

We start with a rather ordinary lens design that shows a image illuminance as calculated above.

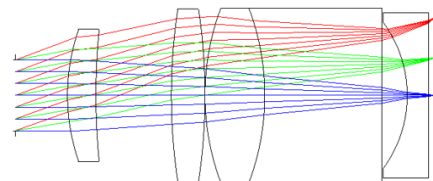


Fig. 2 Lens 1

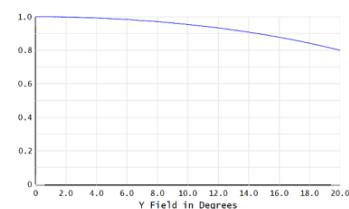
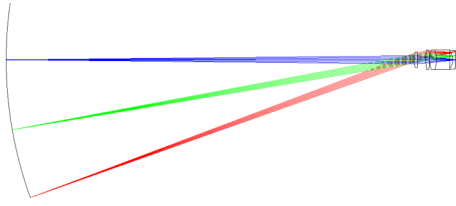
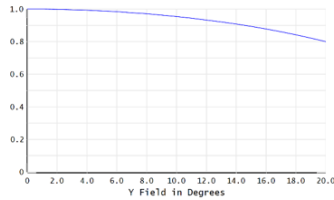


Fig. 3 Relative illumination

We try to get rid of the  $\cos^4$  shape by imaging a curved object with almost the same lens. This way, three of the four cosine factors should disappear - but obviously they don't :

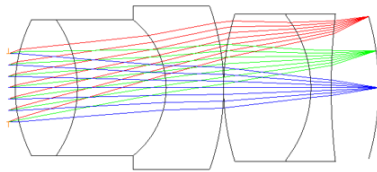


**Fig. 4** Lens 1 imaging a curved object

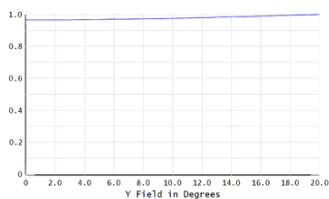


**Fig. 5** Relative illumination

We additionally try a curved image plane. This should not change anything as according to the theory the image side chief ray angle is not involved - but it does :



**Fig. 6** Lens 3

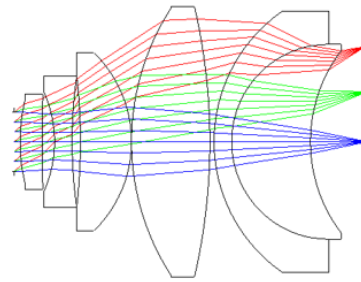


**Fig. 7** Relative illumination

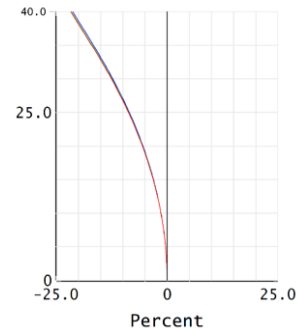
## 4 Outlook

A closer look into the literature (for an introduction see [2, 3]) shows considerable effects of pupil aberrations [4] and distortion [5].

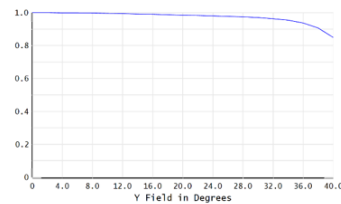
For example, the introduction of some barrel distortion enables an almost homogeneous illumination.



**Fig. 8** Lens 3



**Fig. 9** Distortion of the design



**Fig. 10** Relative illumination

There seems to be much work to do until we can reliably apply such aberrations for some advanced design.

## References

- [1] H. Rehn, "How to fill a light tube," in *Illumination Optics VII*, **13022**, 56–59, SPIE (2024).
- [2] M. v. Rohr, "Über die Lichtverteilung in der Brennebene photographischer Objektive mit besonderer Berücksichtigung der bei einfachen Landschaftslinsen und symmetrischen Konstruktionen auftretenden Unterschiede," *Zeitschrift für Instrumentenkunde* **18** (1898).
- [3] E. Wandersleb, *Die Lichtverteilung im Grossen in der Brennebene des Photographischen Objektivs*, Scientia Photographica, Akademie-Verlag (1952).
- [4] M. Reiss, "The Cos<sup>4</sup> Law of Illumination," *Journal of the Optical Society of America* **35**, 283 (1945).
- [5] R. Siew, "Relative illumination and image distortion," *Optical Engineering* **56**, 049701 (2017).